Differentiability of supremum type functionals: Applications to statistics

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The supremum norm has been used in statistics in a wide variety of situations. One of the best known is the goodness-of-fit Kolmogorov-Smirnov test, where the supremum norm is utilized to measure the discrepancy between the distribution functions.

Let $\mathfrak{X}$ be a non-empty set (usually $\mathbb{R}$, $\overline{\mathbb{R}}^d$ with $\overline{\mathbb{R}} = [-\infty, \infty]$, or $\mathcal{F}$ a class of functions) and $\ell^\infty(\mathfrak{X})$ the space of bounded functions $f : \mathfrak{X} \longrightarrow \mathbb{R}$ equipped with the supremum norm $\|f\|_\infty = \sup\limits_{x\in\mathfrak{X}} (|f(x)|)$. We denote for $f \in \ell^\infty(\mathfrak{X})$

$$\delta(f) = \sup_{x\in\mathfrak{X}} (|f(x)|) \quad \sigma(f) = \sup_{x\in\mathfrak{X}} (f(x)).$$

Suppose that we wish to estimate $\varphi(q)$ for $q \in \ell^\infty(\mathfrak{X})$ and $\varphi \in \{\delta, \sigma\}$. If there exists $\{\mathcal{Q}_n\}_{n\in\mathbb{N}} \subset \ell^\infty(\mathfrak{X})$ and $\{r_n\}_{n\in\mathbb{N}}$ sequence of real numbers such that $r_n \longrightarrow \infty$ and

$$r_n \ (\mathcal{Q}_n - q) \rightsquigarrow \mathcal{Q}$$

as $n \longrightarrow \infty$, where $\rightsquigarrow$ denotes weak convergence and $\mathcal{Q}$ is a tight element of $\ell^\infty(\mathfrak{X})$, it seems natural to use $\varphi(\mathcal{Q}_n)$ to estimate $\varphi(q)$. In other words, if q can be estimated in $\ell^\infty(\mathfrak{X})$ by $\mathcal{Q}_n$, it is reasonable to use the plug-in estimator to approximate $\varphi(q)$. The aim of this work is dealing with the asymptotics of

$$D_n(\varphi) = r_n \ (\varphi(\mathcal{Q}_n) - \varphi(q)),$$

with $\varphi \in \{\delta, \sigma\}$.

To the best of our knowledge, the first remarkable result in this direction was obtained by [4]. The proofs provided in [4] are essentially based on a careful analysis of the behaviour of the empirical process. However, we explore an alternative and more general approach: the Functional Delta Method. In this work we are going to analyze the Hadamard directional differentiability, which in some sense is the most general notion of differentiability in order to apply the Functional Delta Method (see [5]). As particular examples of this framework we show:

- An extension of the results in [4].
- Solution of an open question about Berk-Jones statistic (see [3, Question 2, p. 329]).
- A result about the asymptotic copula process under the alternative.
- Homogeneity test based on kernel distances. Asymptotics results under parameter estimation and supremum kernel distances ([1] and [2])
- On the uniqueness of the set of k-means (under development).

Literatur

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